

5. Probability

- A. Introduction
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Probability is an important and complex field of study. Fortunately, only a few basic issues in probability theory are essential for understanding statistics at the level covered in this book. These basic issues are covered in this chapter.

The introductory section discusses the definitions of probability. This is not as simple as it may seem. The section on basic concepts covers how to compute probabilities in a variety of simple situations. The section on base rates discusses an important but often-ignored factor in determining probabilities.

Remarks on the Concept of “Probability”

by Dan Osherson

Prerequisites

- None

Learning Objectives

1. Define symmetrical outcomes
2. Distinguish between frequentist and subjective approaches
3. Determine whether the frequentist or subjective approach is better suited for a given situation

Inferential statistics is built on the foundation of probability theory, and has been remarkably successful in guiding opinion about the conclusions to be drawn from data. Yet (paradoxically) the very idea of probability has been plagued by controversy from the beginning of the subject to the present day. In this section we provide a glimpse of the debate about the interpretation of the probability concept.

One conception of probability is drawn from the idea of **symmetrical outcomes**. For example, the two possible outcomes of tossing a fair coin seem not to be distinguishable in any way that affects which side will land up or down. Therefore the probability of heads is taken to be $1/2$, as is the probability of tails. In general, if there are N symmetrical outcomes, the probability of any given one of them occurring is taken to be $1/N$. Thus, if a six-sided die is rolled, the probability of any one of the six sides coming up is $1/6$.

Probabilities can also be thought of in terms of **relative frequencies**. If we tossed a coin millions of times, we would expect the proportion of tosses that came up heads to be pretty close to $1/2$. As the number of tosses increases, the proportion of heads approaches $1/2$. Therefore, we can say that the probability of a head is $1/2$.

If it has rained in Seattle on 62% of the last 100,000 days, then the probability of it raining tomorrow might be taken to be 0.62. This is a natural idea but nonetheless unreasonable if we have further information relevant to whether it will rain tomorrow. For example, if tomorrow is August 1, a day of the year on which it seldom rains in Seattle, we should only consider the percentage of the time it rained on August 1. But even this is not enough since the probability of rain on the next August 1 depends on the humidity. (The chances are higher in the presence of high humidity.) So, we should consult only the prior occurrences of

August 1 that had the same humidity as the next occurrence of August 1. Of course, wind direction also affects probability. You can see that our sample of prior cases will soon be reduced to the empty set. Anyway, past meteorological history is misleading if the climate is changing.

For some purposes, probability is best thought of as subjective. Questions such as “What is the probability that Ms. Garcia will defeat Mr. Smith in an upcoming congressional election?” do not conveniently fit into either the symmetry or frequency approaches to probability. Rather, assigning probability 0.7 (say) to this event seems to reflect the speaker's personal opinion --- perhaps his willingness to bet according to certain odds. Such an approach to probability, however, seems to lose the objective content of the idea of chance; probability becomes mere opinion.

Two people might attach different probabilities to the election outcome, yet there would be no criterion for calling one “right” and the other “wrong.” We cannot call one of the two people right simply because she assigned higher probability to the outcome that actually transpires. After all, you would be right to attribute probability $1/6$ to throwing a six with a fair die, and your friend who attributes $2/3$ to this event would be wrong. And you are still right (and your friend is still wrong) even if the die ends up showing a six! The lack of objective criteria for adjudicating claims about probabilities in the subjective perspective is an unattractive feature of it for many scholars.

Like most work in the field, the present text adopts the frequentist approach to probability in most cases. Moreover, almost all the probabilities we shall encounter will be nondogmatic, that is, neither zero nor one. An event with probability 0 has no chance of occurring; an event of probability 1 is certain to occur. It is hard to think of any examples of interest to statistics in which the probability is either 0 or 1. (Even the probability that the Sun will come up tomorrow is less than 1.)

The following example illustrates our attitude about probabilities. Suppose you wish to know what the weather will be like next Saturday because you are planning a picnic. You turn on your radio, and the weather person says, “There is a 10% chance of rain.” You decide to have the picnic outdoors and, lo and behold, it rains. You are furious with the weather person. But was she wrong? No, she did not say it would not rain, only that rain was unlikely. She would have been flatly wrong only if she said that the probability is 0 and it subsequently rained.

However, if you kept track of her weather predictions over a long period of time and found that it rained on 50% of the days that the weather person said the probability was 0.10, you could say her probability assessments are wrong.

So when is it accurate to say that the probability of rain is 0.10? According to our frequency interpretation, it means that it will rain 10% of the days on which rain is forecast with this probability.

Basic Concepts

by David M. Lane

Prerequisites

- Chapter 5: Introduction to Probability

Learning Objectives

1. Compute probability in a situation where there are equally-likely outcomes
2. Apply concepts to cards and dice
3. Compute the probability of two independent events both occurring
4. Compute the probability of either of two independent events occurring
5. Do problems that involve conditional probabilities
6. Compute the probability that in a room of N people, at least two share a birthday
7. Describe the gambler's fallacy

Probability of a Single Event

If you roll a six-sided die, there are six possible outcomes, and each of these outcomes is equally likely. A six is as likely to come up as a three, and likewise for the other four sides of the die. What, then, is the probability that a one will come up? Since there are six possible outcomes, the probability is 1/6. What is the probability that either a one or a six will come up? The two outcomes about which we are concerned (a one or a six coming up) are called favorable outcomes. Given that all outcomes are equally likely, we can compute the probability of a one or a six using the formula:

$$probability = \frac{\text{Number of favorable outcomes}}{\text{Number of possible equally – likely outcomes}}$$

In this case there are two favorable outcomes and six possible outcomes. So the probability of throwing either a one or six is 1/3. Don't be misled by our use of the term “favorable,” by the way. You should understand it in the sense of “favorable to the event in question happening.” That event might not be favorable to your well-being. You might be betting on a three, for example.

The above formula applies to many games of chance. For example, what is the probability that a card drawn at random from a deck of playing cards will be an ace? Since the deck has four aces, there are four favorable outcomes; since the deck has 52 cards, there are 52 possible outcomes. The probability is therefore $4/52 = 1/13$. What about the probability that the card will be a club? Since there are 13 clubs, the probability is $13/52 = 1/4$.

Let's say you have a bag with 20 cherries: 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet? There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is $14/20 = 7/10$. There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

Here is a more complex example. You throw 2 dice. What is the probability that the sum of the two dice will be 6? To solve this problem, list all the possible outcomes. There are 36 of them since each die can come up one of six ways. The 36 possibilities are shown in Table 1.

Table 1. 36 possible outcomes.

Die 1	Die 2	Total	Die 1	Die 2	Total	Die 1	Die 2	Total
1	1	2	3	1	4	5	1	6
1	2	3	3	2	5	5	2	7
1	3	4	3	3	6	5	3	8
1	4	5	3	4	7	5	4	9
1	5	6	3	5	8	5	5	10
1	6	7	3	6	9	5	6	11
2	1	3	4	1	5	6	1	7
2	2	4	4	2	6	6	2	8
2	3	5	4	3	7	6	3	9
2	4	6	4	4	8	6	4	10
2	5	7	4	5	9	6	5	11
2	6	8	4	6	10	6	6	12

You can see that 5 of the 36 possibilities total 6. Therefore, the probability is $5/36$.

If you know the probability of an event occurring, it is easy to compute the probability that the event does not occur. If $P(A)$ is the probability of Event A, then $1 - P(A)$ is the probability that the event does not occur. For the last example, the probability that the total is 6 is $5/36$. Therefore, the probability that the total is not 6 is $1 - 5/36 = 31/36$.

Probability of Two (or more) Independent Events

Events A and B are independent events if the probability of Event B occurring is the same whether or not Event A occurs. Let's take a simple example. A fair coin is tossed two times. The probability that a head comes up on the second toss is $1/2$ regardless of whether or not a head came up on the first toss. The two events are (1) first toss is a head and (2) second toss is a head. So these events are independent. Consider the two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston). These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.

Probability of A and B

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events. More formally, if events A and B are independent, then the probability of both A and B occurring is:

$$P(A \text{ and } B) = P(A) \times P(B)$$

where $P(A \text{ and } B)$ is the probability of events A and B both occurring, $P(A)$ is the probability of event A occurring, and $P(B)$ is the probability of event B occurring.

If you flip a coin twice, what is the probability that it will come up heads both times? Event A is that the coin comes up heads on the first flip and Event B is that the coin comes up heads on the second flip. Since both $P(A)$ and $P(B)$ equal $1/2$, the probability that both events occur is

$$1/2 \times 1/2 = 1/4$$

Let's take another example. If you flip a coin and roll a six-sided die, what is the probability that the coin comes up heads and the die comes up 1? Since the two events are independent, the probability is simply the probability of a head (which is $1/2$) times the probability of the die coming up 1 (which is $1/6$). Therefore, the probability of both events occurring is $1/2 \times 1/6 = 1/12$.

One final example: You draw a card from a deck of cards, put it back, and then draw another card. What is the probability that the first card is a heart and the second card is black? Since there are 52 cards in a deck and 13 of them are hearts, the probability that the first card is a heart is $13/52 = 1/4$. Since there are 26 black cards in the deck, the probability that the second card is black is $26/52 = 1/2$. The probability of both events occurring is therefore $1/4 \times 1/2 = 1/8$.

See the discussion on conditional probabilities on this page to see how to compute $P(A \text{ and } B)$ when A and B are not independent.

Probability of A or B

If Events A and B are independent, the probability that either Event A or Event B occurs is:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

In this discussion, when we say "A or B occurs" we include three possibilities:

1. A occurs and B does not occur
2. B occurs and A does not occur
3. Both A and B occur

This use of the word “or” is technically called inclusive or because it includes the case in which both A and B occur. If we included only the first two cases, then we would be using an exclusive or.

(Optional) We can derive the law for $P(A\text{-or-}B)$ from our law about $P(A\text{-and-}B)$. The event “A-or-B” can happen in any of the following ways:

1. A-and-B happens
2. A-and-not-B happens
3. not-A-and-B happens.

The simple event A can happen if either A-and-B happens or A-and-not-B happens. Similarly, the simple event B happens if either A-and-B happens or not-A-and-B happens. $P(A) + P(B)$ is therefore $P(A\text{-and-}B) + P(A\text{-and-not-}B) + P(A\text{-and-}B) + P(\text{not-A-and-}B)$, whereas $P(A\text{-or-}B)$ is $P(A\text{-and-}B) + P(A\text{-and-not-}B) + P(\text{not-A-and-}B)$. We can make these two sums equal by subtracting one occurrence of $P(A\text{-and-}B)$ from the first. Hence, $P(A\text{-or-}B) = P(A) + P(B) - P(A\text{-and-}B)$.

Now for some examples. If you flip a coin two times, what is the probability that you will get a head on the first flip or a head on the second flip (or both)? Letting Event A be a head on the first flip and Event B be a head on the second flip, then $P(A) = 1/2$, $P(B) = 1/2$, and $P(A \text{ and } B) = 1/4$. Therefore,

$$P(A \text{ or } B) = 1/2 + 1/2 - 1/4 = 3/4.$$

If you throw a six-sided die and then flip a coin, what is the probability that you will get either a 6 on the die or a head on the coin flip (or both)? Using the formula,

$$\begin{aligned} P(6 \text{ or head}) &= P(6) + P(\text{head}) - P(6 \text{ and head}) \\ &= (1/6) + (1/2) - (1/6)(1/2) \\ &= 7/12 \end{aligned}$$

An alternate approach to computing this value is to start by computing the probability of not getting either a 6 or a head. Then subtract this value from 1 to compute the probability of getting a 6 or a head. Although this is a complicated

method, it has the advantage of being applicable to problems with more than two events. Here is the calculation in the present case. The probability of not getting either a 6 or a head can be recast as the probability of

(not getting a 6) AND (not getting a head) .

This follows because if you did not get a 6 and you did not get a head, then you did not get a 6 or a head. The probability of not getting a six is $1 - 1/6 = 5/6$. The probability of not getting a head is $1 - 1/2 = 1/2$. The probability of not getting a six and not getting a head is $5/6 \times 1/2 = 5/12$. This is therefore the probability of not getting a 6 or a head. The probability of getting a six or a head is therefore (once again) $1 - 5/12 = 7/12$.

If you throw a die three times, what is the probability that one or more of your throws will come up with a 1? That is, what is the probability of getting a 1 on the first throw OR a 1 on the second throw OR a 1 on the third throw? The easiest way to approach this problem is to compute the probability of

NOT getting a 1 on the first throw
AND not getting a 1 on the second throw
AND not getting a 1 on the third throw.

The answer will be 1 minus this probability. The probability of not getting a 1 on any of the three throws is $5/6 \times 5/6 \times 5/6 = 125/216$. Therefore, the probability of getting a 1 on at least one of the throws is $1 - 125/216 = 91/216$.

Conditional Probabilities

Often it is required to compute the probability of an event given that another event has occurred. For example, what is the probability that two cards drawn at random from a deck of playing cards will both be aces? It might seem that you could use the formula for the probability of two independent events and simply multiply $4/52 \times 4/52 = 1/169$. This would be **incorrect**, however, because the two events are not independent. If the first card drawn is an ace, then the probability that the second card is also an ace would be lower because there would only be three aces left in the deck.

Once the first card chosen is an ace, the probability that the second card chosen is also an ace is called the conditional probability of drawing an ace. In this case, the “condition” is that the first card is an ace. Symbolically, we write this as:

$$P(\text{ace on second draw} \mid \text{an ace on the first draw})$$

The vertical bar “|” is read as “given,” so the above expression is short for: “The probability that an ace is drawn on the second draw given that an ace was drawn on the first draw.” What is this probability? Since after an ace is drawn on the first draw, there are 3 aces out of 51 total cards left. This means that the probability that one of these aces will be drawn is $3/51 = 1/17$.

If Events A and B are not independent, then
 $P(A \text{ and } B) = P(A) \times P(B|A)$.

Applying this to the problem of two aces, the probability of drawing two aces from a deck is $4/52 \times 3/51 = 1/221$.

One more example: If you draw two cards from a deck, what is the probability that you will get the Ace of Diamonds and a black card? There are two ways you can satisfy this condition: (a) You can get the Ace of Diamonds first and then a black card or (b) you can get a black card first and then the Ace of Diamonds. Let's calculate Case A. The probability that the first card is the Ace of Diamonds is $1/52$. The probability that the second card is black given that the first card is the Ace of Diamonds is $26/51$ because 26 of the remaining 51 cards are black. The probability is therefore $1/52 \times 26/51 = 1/102$. Now for Case B: the probability that the first card is black is $26/52 = 1/2$. The probability that the second card is the Ace of Diamonds given that the first card is black is $1/51$. The probability of Case B is therefore $1/2 \times 1/51 = 1/102$, the same as the probability of Case A. Recall that the probability of A or B is $P(A) + P(B) - P(A \text{ and } B)$. In this problem, $P(A \text{ and } B) = 0$ since a card cannot be the Ace of Diamonds and be a black card. Therefore, the probability of Case A or Case B is $1/102 + 1/102 = 2/102 = 1/51$. So, $1/51$ is the probability that you will get the Ace of Diamonds and a black card when drawing two cards from a deck.

Birthday Problem

If there are 25 people in a room, what is the probability that at least two of them share the same birthday. If your first thought is that it is $25/365 = 0.068$, you will be surprised to learn it is much higher than that. This problem requires the application of the sections on $P(A \text{ and } B)$ and conditional probability.

This problem is best approached by asking what is the probability that no two people have the same birthday. Once we know this probability, we can simply subtract it from 1 to find the probability that two people share a birthday.

If we choose two people at random, what is the probability that they do not share a birthday? Of the 365 days on which the second person could have a birthday, 364 of them are different from the first person's birthday. Therefore the probability is $364/365$. Let's define P_2 as the probability that the second person drawn does not share a birthday with the person drawn previously. P_2 is therefore $364/365$. Now define P_3 as the probability that the third person drawn does not share a birthday with anyone drawn previously **given** that there are no previous birthday matches. P_3 is therefore a conditional probability. If there are no previous birthday matches, then two of the 365 days have been "used up," leaving 363 non-matching days. Therefore $P_3 = 363/365$. In like manner, $P_4 = 362/365$, $P_5 = 361/365$, and so on up to $P_{25} = 341/365$.

In order for there to be no matches, the second person must not match any previous person **and** the third person must not match any previous person, and the fourth person must not match any previous person, etc. Since $P(A \text{ and } B) = P(A)P(B)$, all we have to do is multiply $P_2, P_3, P_4 \dots P_{25}$ together. The result is 0.431. Therefore the probability of at least one match is 0.569.

Gambler's Fallacy

A fair coin is flipped five times and comes up heads each time. What is the probability that it will come up heads on the sixth flip? The correct answer is, of course, $1/2$. But many people believe that a tail is more likely to occur after throwing five heads. Their **faulty reasoning** may go something like this: "In the long run, the number of heads and tails will be the same, so the tails have some catching up to do."

The error in this reasoning is that the proportion of heads approaches 0.5 but the number of heads does not approach the number of tails. The results of a simulation ([external link](#); requires Java) are shown in Figure 1. (The quality of the image is somewhat low because it was captured from the screen.)

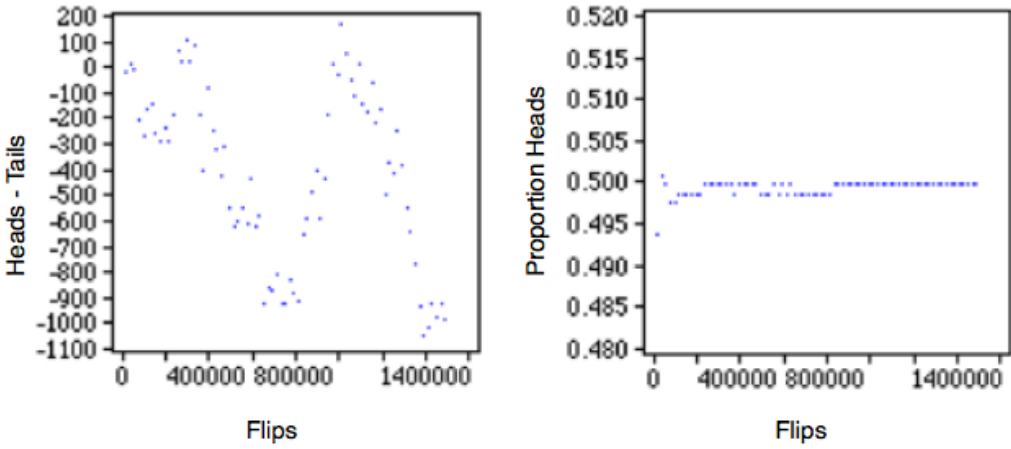


Figure 1. The results of simulating 1,500,000 coin flips. The graph on the left shows the difference between the number of heads and the number of tails as a function of the number of flips. You can see that there is no consistent pattern. After the final flip, there are 968 more tails than heads. The graph on the right shows the proportion of heads. This value goes up and down at the beginning, but converges to 0.5 (rounded to 3 decimal places) before 1,000,000 flips.

Permutations and Combinations

by David M. Lane

Prerequisites

none

Learning Objectives

1. Calculate the probability of two independent events occurring
2. Define permutations and combinations
3. List all permutations and combinations
4. Apply formulas for permutations and combinations

This section covers basic formulas for determining the number of various possible types of outcomes. The topics covered are: (1) counting the number of possible orders, (2) counting using the multiplication rule, (3) counting the number of permutations, and (4) counting the number of combinations.

Possible Orders

Suppose you had a plate with three pieces of candy on it: one green, one yellow, and one red. You are going to pick up these three pieces one at a time. The question is: In how many different orders can you pick up the pieces? Table 1 lists all the possible orders. There are two orders in which red is first: red, yellow, green and red, green, yellow. Similarly, there are two orders in which yellow is first and two orders in which green is first. This makes six possible orders in which the pieces can be picked up.

Table 1. Six Possible Orders.

Number	First	Second	Third
1	red	yellow	green
2	red	green	yellow
3	yellow	red	green
4	yellow	green	red
5	green	red	yellow
6	green	yellow	red

The formula for the number of orders is shown below.

$$\text{Number of orders} = n!$$

where n is the number of pieces to be picked up. The symbol “!” stands for factorial. Some examples are:

$$3! = 3 \times 2 \times 1 = 6$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

This means that if there were 5 pieces of candy to be picked up, they could be picked up in any of $5! = 120$ orders.

Multiplication Rule

Imagine a small restaurant whose menu has 3 soups, 6 entrées, and 4 desserts. How many possible meals are there? The answer is calculated by multiplying the numbers to get $3 \times 6 \times 4 = 72$. You can think of it as first there is a choice among 3 soups. Then, for each of these choices there is a choice among 6 entrées resulting in $3 \times 6 = 18$ possibilities. Then, for each of these 18 possibilities there are 4 possible desserts yielding $18 \times 4 = 72$ total possibilities.

Permutations

Suppose that there were four pieces of candy (red, yellow, green, and brown) and you were only going to pick up exactly two pieces. How many ways are there of

picking up two pieces? Table 2 lists all the possibilities. The first choice can be any of the four colors. For each of these 4 first choices there are 3 second choices. Therefore there are $4 \times 3 = 12$ possibilities.

Table 2. Twelve Possible Orders.

Number	First	Second
1	red	yellow
2	red	green
3	red	brown
4	yellow	red
5	yellow	green
6	yellow	brown
7	green	red
8	green	yellow
9	green	brown
10	brown	red
11	brown	yellow
12	brown	green

More formally, this question is asking for the number of permutations of four things taken two at a time. The general formula is:

$${}_n P_r = \frac{n!}{(n - r)!}$$

where ${}_n P_r$ is the number of permutations of n things taken r at a time. In other words, it is the number of ways r things can be selected from a group of n things. In this case,

$${}_4 P_2 = \frac{4!}{(4 - 2)!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 12$$

It is important to note that order counts in permutations. That is, choosing red and then yellow is counted separately from choosing yellow and then red. Therefore permutations refer to the number of ways of choosing rather than the number of

possible outcomes. When order of choice is not considered, the formula for combinations is used.

Combinations

Now suppose that you were not concerned with the way the pieces of candy were chosen but only in the final choices. In other words, how many different combinations of two pieces could you end up with? In counting combinations, choosing red and then yellow is the same as choosing yellow and then red because in both cases you end up with one red piece and one yellow piece. Unlike permutations, order does not count. Table 3 is based on Table 2 but is modified so that repeated combinations are given an “x” instead of a number. For example, “yellow then red” has an “x” because the combination of red and yellow was already included as choice number 1. As you can see, there are six combinations of the three colors.

Table 3. Six Combinations.

Number	First	Second
1	red	yellow
2	red	green
3	red	brown
x	yellow	red
4	yellow	green
5	yellow	brown
x	green	red
x	green	yellow
6	green	brown
x	brown	red
x	brown	yellow
x	brown	green

The formula for the number of combinations is shown below where nCr is the number of combinations for n things taken r at a time.

$${}_n C_r = \frac{n!}{(n-r)! r!}$$

For our example,

$${}_4 C_2 = \frac{4!}{(4-2)! 2!} = \frac{4 \times 3 \times 2 \times 1}{(2 \times 1)(2 \times 1)} = 6$$

which is consistent with Table 3.

As an example application, suppose there were six kinds of toppings that one could order for a pizza. How many combinations of exactly 3 toppings could be ordered? Here $n = 6$ since there are 6 toppings and $r = 3$ since we are taking 3 at a time. The formula is then:

$${}_6 C_3 = \frac{6!}{(6-3)! 3!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(3 \times 2 \times 1)} = 20.$$

Binomial Distribution

by David M. Lane

Prerequisites

- Chapter 1: Distributions
- Chapter 3: Variability
- Chapter 5: Basic Probability

Learning Objectives

1. Define binomial outcomes
2. Compute the probability of getting X successes in N trials
3. Compute cumulative binomial probabilities
4. Find the mean and standard deviation of a binomial distribution

When you flip a coin, there are two possible outcomes: heads and tails. Each outcome has a fixed probability, the same from trial to trial. In the case of coins, heads and tails each have the same probability of $1/2$. More generally, there are situations in which the coin is biased, so that heads and tails have different probabilities. In the present section, we consider probability distributions for which there are just two possible outcomes with fixed probabilities summing to one. These distributions are called binomial distributions.

A Simple Example

The four possible outcomes that could occur if you flipped a coin twice are listed below in Table 1. Note that the four outcomes are equally likely: each has probability $1/4$. To see this, note that the tosses of the coin are independent (neither affects the other). Hence, the probability of a head on Flip 1 and a head on Flip 2 is the product of $P(H)$ and $P(H)$, which is $1/2 \times 1/2 = 1/4$. The same calculation applies to the probability of a head on Flip 1 and a tail on Flip 2. Each is $1/2 \times 1/2 = 1/4$.

Table 1. Four Possible Outcomes.

Outcome	First Flip	Second Flip
1	Heads	Heads
2	Heads	Tails

3	Tails	Heads
4	Tails	Tails

The four possible outcomes can be classified in terms of the number of heads that come up. The number could be two (Outcome 1), one (Outcomes 2 and 3) or 0 (Outcome 4). The probabilities of these possibilities are shown in Table 2 and in Figure 1. Since two of the outcomes represent the case in which just one head appears in the two tosses, the probability of this event is equal to $1/4 + 1/4 = 1/2$. Table 2 summarizes the situation.

Table 2. Probabilities of Getting 0, 1, or 2 Heads.

Number of Heads	Probability
0	1/4
1	1/2
2	1/4

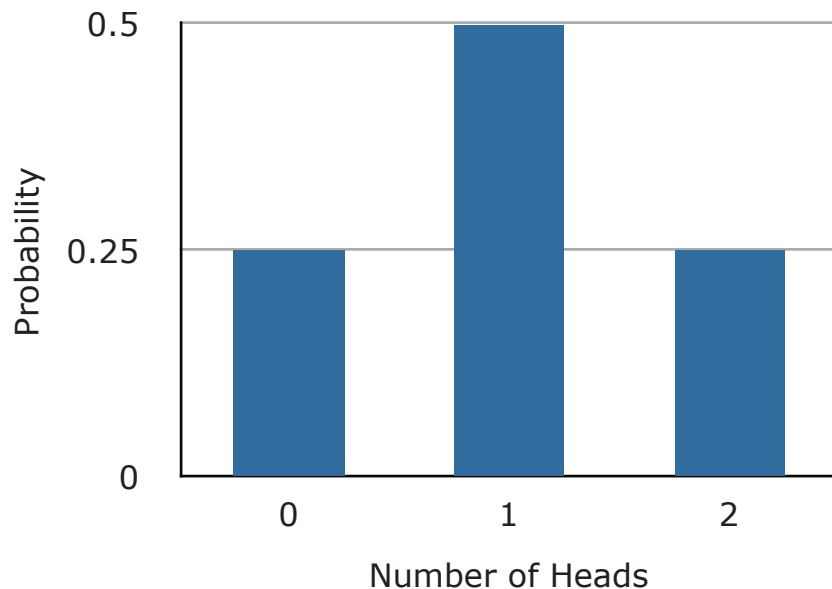


Figure 1. Probabilities of 0, 1, and 2 heads.

Figure 1 is a discrete probability distribution: It shows the probability for each of the values on the X-axis. Defining a head as a “success,” Figure 1 shows the probability of 0, 1, and 2 successes for two trials (flips) for an event that has a

probability of 0.5 of being a success on each trial. This makes Figure 1 an example of a binomial distribution.

The Formula for Binomial Probabilities

The binomial distribution consists of the probabilities of each of the possible numbers of successes on N trials for independent events that each have a probability of π (the Greek letter pi) of occurring. For the coin flip example, $N = 2$ and $\pi = 0.5$. The formula for the binomial distribution is shown below:

$$P(x) = \frac{N!}{x!(N - X)!} \pi^x (1 - \pi)^{N-x}$$

where $P(x)$ is the probability of x successes out of N trials, N is the number of trials, and π is the probability of success on a given trial. Applying this to the coin flip example,

$$P(0) = \frac{2!}{0!(2 - 0)!} (.5^0)(1 - .5)^{2-0} = \frac{2}{2} (1)(.25) = 0.25$$

$$P(1) = \frac{2!}{1!(2 - 1)!} (.5^1)(1 - .5)^{2-1} = \frac{2}{1} (.5)(.5) = 0.50$$

$$P(2) = \frac{2!}{2!(2 - 2)!} (.5^2)(1 - .5)^{2-2} = \frac{2}{2} (.25)(1) = 0.25$$

If you flip a coin twice, what is the probability of getting one or more heads? Since the probability of getting exactly one head is 0.50 and the probability of getting exactly two heads is 0.25, the probability of getting one or more heads is $0.50 + 0.25 = 0.75$.

Now suppose that the coin is biased. The probability of heads is only 0.4. What is the probability of getting heads at least once in two tosses? Substituting into the general formula above, you should obtain the answer .64.

Cumulative Probabilities

We toss a coin 12 times. What is the probability that we get from 0 to 3 heads? The answer is found by computing the probability of exactly 0 heads, exactly 1 head, exactly 2 heads, and exactly 3 heads. The probability of getting from 0 to 3 heads

is then the sum of these probabilities. The probabilities are: 0.0002, 0.0029, 0.0161, and 0.0537. The sum of the probabilities is 0.073. The calculation of cumulative binomial probabilities can be quite tedious. Therefore we have provided a binomial calculator ([external link](#); requires Java) to make it easy to calculate these probabilities.

Mean and Standard Deviation of Binomial Distributions

Consider a coin-tossing experiment in which you tossed a coin 12 times and recorded the number of heads. If you performed this experiment over and over again, what would the mean number of heads be? On average, you would expect half the coin tosses to come up heads. Therefore the mean number of heads would be 6. In general, the mean of a binomial distribution with parameters N (the number of trials) and π (the probability of success on each trial) is:

$$\mu = N\pi$$

where μ is the mean of the binomial distribution. The variance of the binomial distribution is:

$$\sigma^2 = N\pi(1-\pi)$$

where σ^2 is the variance of the binomial distribution.

Let's return to the coin-tossing experiment. The coin was tossed 12 times, so $N = 12$. A coin has a probability of 0.5 of coming up heads. Therefore, $\pi = 0.5$. The mean and variance can therefore be computed as follows:

$$\begin{aligned}\mu &= N\pi = (12)(0.5) = 6 \\ \sigma^2 &= N\pi(1-\pi) = (12)(0.5)(1.0 - 0.5) = 3.0.\end{aligned}$$

Naturally, the standard deviation (σ) is the square root of the variance (σ^2).

$$\sigma = \sqrt{N\pi(1-\pi)}$$

Poisson Distribution

by David M. Lane

Prerequisites

- Chapter 1: Logarithms

The Poisson distribution can be used to calculate the probabilities of various numbers of “successes” based on the mean number of successes. In order to apply the Poisson distribution, the various events must be independent. Keep in mind that the term “success” does not really mean success in the traditional positive sense. It just means that the outcome in question occurs.

Suppose you knew that the mean number of calls to a fire station on a weekday is 8. What is the probability that on a given weekday there would be 11 calls? This problem can be solved using the following formula based on the Poisson distribution:

$$p = \frac{e^{-\mu} \mu^x}{x!}$$

e is the base of natural logarithms (2.7183)

μ is the mean number of “successes”

x is the number of “successes” in question

For this example,

$$p = \frac{e^{-8} 8^{11}}{11!} = 0.072$$

since the mean is 8 and the question pertains to 11 fires.

The mean of the Poisson distribution is μ . The variance is also equal to μ . Thus, for this example, both the mean and the variance are equal to 8.

Multinomial Distribution

by David M. Lane

Prerequisites

- Chapter 1: Distributions
- Chapter 3: Variability
- Chapter 5: Basic Probability
- Chapter 5: Binomial Distribution

Learning Objectives

1. Define multinomial outcomes
2. Compute probabilities using the multinomial distribution

The binomial distribution allows one to compute the probability of obtaining a given number of binary outcomes. For example, it can be used to compute the probability of getting 6 heads out of 10 coin flips. The flip of a coin is a binary outcome because it has only two possible outcomes: heads and tails. The multinomial distribution can be used to compute the probabilities in situations in which there are more than two possible outcomes. For example, suppose that two chess players had played numerous games and it was determined that the probability that Player A would win is 0.40, the probability that Player B would win is 0.35, and the probability that the game would end in a draw is 0.25. The multinomial distribution can be used to answer questions such as: “If these two chess players played 12 games, what is the probability that Player A would win 7 games, Player B would win 2 games, and the remaining 3 games would be drawn?” The following formula gives the probability of obtaining a specific set of outcomes when there are three possible outcomes for each event:

$$p = \frac{n!}{(n_1!)(n_2!)(n_3!)} p_1^{n_1} p_2^{n_2} p_3^{n_3}$$

where

p is the probability,
 n is the total number of events

n_1 is the number of times Outcome 1 occurs,
 n_2 is the number of times Outcome 2 occurs,
 n_3 is the number of times Outcome 3 occurs,
 p_1 is the probability of Outcome 1
 p_2 is the probability of Outcome 2, and
 p_3 is the probability of Outcome 3.

For the chess example,

$n = 12$ (12 games are played),
 $n_1 = 7$ (number won by Player A),
 $n_2 = 2$ (number won by Player B),
 $n_3 = 3$ (the number drawn),
 $p_1 = 0.40$ (probability Player A wins)
 $p_2 = 0.35$ (probability Player B wins)
 $p_3 = 0.25$ (probability of a draw)

$$p = \frac{12!}{(7!)(2!)(3!)} \cdot 40^7 \cdot 35^2 \cdot 25^3 = 0.0248$$

The formula for k outcomes is:

$$p = \frac{n!}{(n_1!)(n_2!) \dots (n_k!)} p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$$

Hypergeometric Distribution

by David M. Lane

Prerequisites

- Chapter 5: Binomial Distribution
- Chapter 5: Permutations and Combinations

The hypergeometric distribution is used to calculate probabilities when sampling without replacement. For example, suppose you first randomly sample one card from a deck of 52. Then, without putting the card back in the deck you sample a second and then (again without replacing cards) a third. Given this sampling procedure, what is the probability that exactly two of the sampled cards will be aces (4 of the 52 cards in the deck are aces). You can calculate this probability using the following formula based on the hypergeometric distribution:

$$p = \frac{{}_k C_x \cdot {}_{(N-k)} C_{(n-x)}}{{}_N C_n}$$

where

`k` is the number of "successes" in the population

`x` is the number of "successes" in the sample

`N` is the size of the population

`n` is the number sampled

`p` is the probability of obtaining exactly `x` successes

`{}_k C_x` is the number of combinations of `k` things taken `x` at a time

In this example, `k = 4` because there are four aces in the deck, `x = 2` because the problem asks about the probability of getting two aces, `N = 52` because there are 52 cards in a deck, and `n = 3` because 3 cards were sampled. Therefore,

$$p = \frac{{}^4C_2 {}^{(52-4)}C_{(3-2)}}{{}_{52}C_3}$$

$$p = \frac{\frac{4!}{2!2!} \frac{48!}{47!1!}}{\frac{52!}{49!3!}} = 0.013$$

The mean and standard deviation of the hypergeometric distribution are:

$$\text{mean} = \frac{(n)(k)}{N}$$

$$\text{sd} = \sqrt{\frac{(n)(k)(N-k)(N-n)}{N^2(N-1)}}$$

Base Rates

by David M. Lane

Prerequisites

- Chapter 5: Basic Concepts

Learning Objectives

1. Compute the probability of a condition from hits, false alarms, and base rates using a tree diagram
2. Compute the probability of a condition from hits, false alarms, and base rates using Bayes' Theorem

Suppose that at your regular physical exam you test positive for Disease X. Although Disease X has only mild symptoms, you are concerned and ask your doctor about the accuracy of the test. It turns out that the test is 95% accurate. It would appear that the probability that you have Disease X is therefore 0.95. However, the situation is not that simple.

For one thing, more information about the accuracy of the test is needed because there are two kinds of errors the test can make: misses and false positives. If you actually have Disease X and the test failed to detect it, that would be a miss. If you did not have Disease X and the test indicated you did, that would be a false positive. The miss and false positive rates are not necessarily the same. For example, suppose that the test accurately indicates the disease in 99% of the people who have it and accurately indicates no disease in 91% of the people who do not have it. In other words, the test has a miss rate of 0.01 and a false positive rate of 0.09. This might lead you to revise your judgment and conclude that your chance of having the disease is 0.91. This would not be correct since the probability depends on the proportion of people having the disease. This proportion is called the base rate.

Assume that Disease X is a rare disease, and only 2% of people in your situation have it. How does that affect the probability that you have it? Or, more generally, what is the probability that someone who tests positive actually has the disease? Let's consider what would happen if one million people were tested. Out of these one million people, 2% or 20,000 people would have the disease. Of these 20,000 with the disease, the test would accurately detect it in 99% of them. This means that 19,800 cases would be accurately identified. Now let's consider the

98% of the one million people (980,000) who do not have the disease. Since the false positive rate is 0.09, 9% of these 980,000 people will test positive for the disease. This is a total of 88,200 people incorrectly diagnosed.

To sum up, 19,800 people who tested positive would actually have the disease and 88,200 people who tested positive would not have the disease. This means that of all those who tested positive, only

$$19,800 / (19,800 + 88,200) = 0.1833$$

of them would actually have the disease. So the probability that you have the disease is not 0.95, or 0.91, but only 0.1833.

These results are summarized in Table 1. The numbers of people diagnosed with the disease are shown in red. Of the one million people tested, the test was correct for 891,800 of those without the disease and for 19,800 with the disease; the test was correct 91% of the time. However, if you look only at the people testing positive (shown in red), only 19,800 (0.1833) of the 88,200 + 19,800 = 108,000 testing positive actually have the disease.

Table 1. Diagnosing Disease X.

True Condition			
No Disease 980,000		Disease 20,000	
Test Result		Test Result	
Positive 88,200	Negative 891,800	Positive 19,800	Negative 200

Bayes' Theorem

This same result can be obtained using Bayes' theorem. Bayes' theorem considers both the prior probability of an event and the diagnostic value of a test to determine the posterior probability of the event. For the current example, the event is that you have Disease X. Let's call this Event D. Since only 2% of people in your situation have Disease X, the prior probability of Event D is 0.02. Or, more formally, $P(D) = 0.02$. If D' represents the probability that Event D is false, then $P(D') = 1 - P(D) = 0.98$.

To define the diagnostic value of the test, we need to define another event: that you test positive for Disease X. Let's call this Event T. The diagnostic value of the test depends on the probability you will test positive given that you actually have the disease, written as $P(T|D)$, and the probability you test positive given that you do not have the disease, written as $P(T|D')$. Bayes' theorem shown below allows you to calculate $P(D|T)$, the probability that you have the disease given that you test positive for it.

$$P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D')P(D')}$$

The various terms are:

$$\begin{aligned} P(T|D) &= 0.99 \\ P(T|D') &= 0.09 \\ P(D) &= 0.02 \\ P(D') &= 0.98 \end{aligned}$$

Therefore,

$$P(D|T) = \frac{(0.99)(0.02)}{(0.99)(0.02) + (0.09)(0.98)} = 0.1833$$

which is the same value computed previously.

Statistical Literacy

by David M. Lane

Prerequisites

- Chapter 5: Base Rates

This [webpage](#) gives the FBI list of warning signs for school shooters.

What do you think?

Do you think it is likely that someone showing a majority of these signs would actually shoot people in school?

Fortunately the vast majority of students do not become shooters. It is necessary to take this base rate information into account in order to compute the probability that any given student will be a shooter. The warning signs are unlikely to be sufficiently predictive to warrant the conclusion that a student will become a shooter. If an action is taken on the basis of these warning signs, it is likely that the student involved would never have become a shooter even without the action.

Exercises

Prerequisites

- All material presented in the Probability Chapter

1. (a) What is the probability of rolling a pair of dice and obtaining a total score of 9 or more? (b) What is the probability of rolling a pair of dice and obtaining a total score of 7?
2. A box contains four black pieces of cloth, two striped pieces, and six dotted pieces. A piece is selected randomly and then placed back in the box. A second piece is selected randomly. What is the probability that:
 - a. both pieces are dotted?
 - b. the first piece is black and the second piece is dotted?
 - c. one piece is black and one piece is striped?
3. A card is drawn at random from a deck. (a) What is the probability that it is an ace or a king? (b) What is the probability that it is either a red card or a black card?
4. The probability that you will win a game is 0.45. (a) If you play the game 80 times, what is the most likely number of wins? (b) What are the mean and variance of a binomial distribution with $p = 0.45$ and $N = 80$?
5. A fair coin is flipped 9 times. What is the probability of getting exactly 6 heads?
6. When Susan and Jessica play a card game, Susan wins 60% of the time. If they play 9 games, what is the probability that Jessica will have won more games than Susan?
7. You flip a coin three times. (a) What is the probability of getting heads on only one of your flips? (b) What is the probability of getting heads on at least one flip?
8. A test correctly identifies a disease in 95% of people who have it. It correctly identifies no disease in 94% of people who do not have it. In the population, 3% of the people have the disease. What is the probability that you have the disease if you tested positive?

9. A jar contains 10 blue marbles, 5 red marbles, 4 green marbles, and 1 yellow marble. Two marbles are chosen (without replacement). (a) What is the probability that one will be green and the other red? (b) What is the probability that one will be blue and the other yellow?
10. You roll a fair die five times, and you get a 6 each time. What is the probability that you get a 6 on the next roll?
11. You win a game if you roll a die and get a 2 or a 5. You play this game 60 times.
- What is the probability that you win between 5 and 10 times (inclusive)?
 - What is the probability that you will win the game at least 15 times?
 - What is the probability that you will win the game at least 40 times?
 - What is the most likely number of wins.
 - What is the probability of obtaining the number of wins in d?
- Explain how you got each answer or show your work.
12. In a baseball game, Tommy gets a hit 30% of the time when facing this pitcher. Joey gets a hit 25% of the time. They are both coming up to bat this inning.
- What is the probability that Joey or Tommy will get a hit?
 - What is the probability that neither player gets a hit?
 - What is the probability that they both get a hit?
13. An unfair coin has a probability of coming up heads of 0.65. The coin is flipped 50 times. What is the probability it will come up heads 25 or fewer times? (Give answer to at least 3 decimal places).
14. You draw two cards from a deck, what is the probability that:
- both of them are face cards (king, queen, or jack)?
 - you draw two cards from a deck and both of them are hearts?
15. True/False: You are more likely to get a pattern of HTHHHTHTTH than HHHHHHHHTT when you flip a coin 10 times.

16. True/False: Suppose that at your regular physical exam you test positive for a relatively rare disease. You will need to start taking medicine if you have the disease, so you ask your doctor about the accuracy of the test. It turns out that the test is 98% accurate. The probability that you have Disease X is therefore 0.98 and the probability that you do not have it is .02. Explain your answer.

Questions from Case Studies

Diet and Health (DH) case study

17. (DH)

- a. What percentage of people on the AHA diet had some sort of illness or death?
- b. What is the probability that if you randomly selected a person on the AHA diet, he or she would have some sort of illness or death?
- c. If 3 people on the AHA diet are chosen at random, what is the probability that they will all be healthy?

18. (DH)

- a. What percentage of people on the Mediterranean diet had some sort of illness or death?
- b. What is the probability that if you randomly selected a person on the Mediterranean diet, he or she would have some sort of illness or death?
- c. What is the probability that if you randomly selected a person on the Mediterranean diet, he or she would have cancer?
- d. If you randomly select five people from the Mediterranean diet, what is the probability that they would all be healthy?

The following questions are from ARTIST (reproduced with permission)



19. Five faces of a fair die are painted black, and one face is painted white. The die is rolled six times. Which of the following results is more likely?
- Black side up on five of the rolls; white side up on the other roll
 - Black side up on all six rolls
 - a and b are equally likely
20. One of the items on the student survey for an introductory statistics course was “Rate your intelligence on a scale of 1 to 10.” The distribution of this variable for the 100 women in the class is presented below. What is the probability of randomly selecting a women from the class who has an intelligence rating that is LESS than seven (7)?

Intelligence Rating	Count
5	12
6	24
7	38
8	23
9	2
10	1

- $(12 + 24)/100 = .36$
 - $(12 + 24 + 38)/100 = .74$
 - $38/100 = .38$
 - $(23 + 2 + 1)/100 = .26$
 - None of the above.
21. You roll 2 fair six-sided dice. Which of the following outcomes is most likely to occur on the next roll? A. Getting double 3. B. Getting a 3 and a 4. C. They are equally likely. Explain your choice.

22. If Tahnee flips a coin 10 times, and records the results (Heads or Tails), which outcome below is more likely to occur, A or B? Explain your choice.

Throw Number	1	2	3	4	5	6	7	8	9	10
A	H	T	T	H	T	H	H	T	T	T
B	H	T	H	T	H	T	H	T	H	T

23. A bowl has 100 wrapped hard candies in it. 20 are yellow, 50 are red, and 30 are blue. They are well mixed up in the bowl. Jenny pulls out a handful of 10 candies, counts the number of reds, and tells her teacher. The teacher writes the number of red candies on a list. Then, Jenny puts the candies back into the bowl, and mixes them all up again. Four of Jenny's classmates, Jack, Julie, Jason, and Jerry do the same thing. They each pick ten candies, count the reds, and the teacher writes down the number of reds. Then they put the candies back and mix them up again each time. The teacher's list for the number of reds is most likely to be (please select one):

- a. 8,9,7,10,9
- b. 3,7,5,8,5
- c. 5,5,5,5,5
- d. 2,4,3,4,3
- e. 3,0,9,2,8

24. An insurance company writes policies for a large number of newly-licensed drivers each year. Suppose 40% of these are low-risk drivers, 40% are moderate risk, and 20% are high risk. The company has no way to know which group any individual driver falls in when it writes the policies. None of the low-risk drivers will have an at-fault accident in the next year, but 10% of the moderate-risk and 20% of the high-risk drivers will have such an accident. If a driver has an at-fault accident in the next year, what is the probability that he or she is high-risk?

25. You are to participate in an exam for which you had no chance to study, and for that reason cannot do anything but guess for each question (all questions being of the multiple choice type, so the chance of guessing the correct answer for each question is $1/d$, d being the number of options (distractors) per question;

so in case of a 4-choice question, your guess chance is 0.25). Your instructor offers you the opportunity to choose amongst the following exam formats: I. 6 questions of the 4-choice type; you pass when 5 or more answers are correct; II. 5 questions of the 5-choice type; you pass when 4 or more answers are correct; III. 4 questions of the 10-choice type; you pass when 3 or more answers are correct. Rank the three exam formats according to their attractiveness. It should be clear that the format with the highest probability to pass is the most attractive format. Which would you choose and why?

26. Consider the question of whether the home team wins more than half of its games in the National Basketball Association. Suppose that you study a simple random sample of 80 professional basketball games and find that 52 of them are won by the home team.
- Assuming that there is no home court advantage and that the home team therefore wins 50% of its games in the long run, determine the probability that the home team would win 65% or more of its games in a simple random sample of 80 games.
 - Does the sample information (that 52 of a random sample of 80 games are won by the home team) provide strong evidence that the home team wins more than half of its games in the long run? Explain.
27. A refrigerator contains 6 apples, 5 oranges, 10 bananas, 3 pears, 7 peaches, 11 plums, and 2 mangos.
- Imagine you stick your hand in this refrigerator and pull out a piece of fruit at random. What is the probability that you will pull out a pear?
 - Imagine now that you put your hand in the refrigerator and pull out a piece of fruit. You decide you do not want to eat that fruit so you put it back into the refrigerator and pull out another piece of fruit. What is the probability that the first piece of fruit you pull out is a banana and the second piece you pull out is an apple?
 - What is the probability that you stick your hand in the refrigerator one time and pull out a mango or an orange?